Name Mohammad Khalaf, ID 34709

Linear Algebra MTH 221 Fall 2010, 1-4

Exam I , MTH 221 , Fall 2010, 11 am section

Ayman Badawi

96

QUESTION 1. (30 points) Given $A^{-1} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 2 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ -4 & 2 & 2 \\ 0 & 1 & -2 \end{bmatrix}$

(i) Find the 2nd column of $(AB)^{-1}$ $= B^{-1}A^{-1}$

$$-2\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} -12 \\ 18 \\ -1 \end{bmatrix}}$$

(-2, 8; -6, 6; -4, 4)

(ii) Find the third row of $(BA)^{-1}$ $= A^{-1} \cdot B^{-1}$

$$-2\begin{bmatrix} 1 & -2 & -2 \end{bmatrix} + 2\begin{bmatrix} -4 & 2 & 2 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}$$

(-2, 4, 4; -8, 4, 4; 0, 3, -6)

$$= \boxed{\begin{bmatrix} -10 & 11 & 2 \end{bmatrix}}$$

(iii) Find the (1, 3)-entry $B^{-1}A^{-1}$

1st row of B^{-1} X 3rd colomn of A^{-1}

$$\begin{bmatrix} 1 & -2 & -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = (-2 \times 1) + (0 \times -2) + (3 \times -2) = \boxed{-8}$$

(-2, -6)

(iv) Solve the system $AX = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$

$$A^{-1}AX = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$4\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} - 1\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -1 \end{bmatrix}$$

(8, -8, 8; 2, -3, -2; -6, 0, 9)

$$X = \begin{bmatrix} 4 \\ -11 \\ -1 \end{bmatrix}$$

(v) Find the matrix A

$$\left[A \mid I_n\right] \xrightarrow{\text{row operations}} \left[I_n \mid A^{-1}\right]$$

I want to make $A^{-1} \to I_n$ so this I_n will give me A

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & -2 \\ 0 & 1 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 2 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 0.5 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 2 & 3 \end{array}\right]$$

$2R_1 + R_2$

$2R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 0.5 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1.5 & 1 & 0 & 1 & 0 & -3 \\ 1 & 1 & 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\to$ behind page

QUESTION 2. (12 points) Let $D = \begin{bmatrix} 4 & 2 & b & a \\ -4 & -2 & -2 & 2 \\ 2 & 0.5b & 0 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

(i) For what values of a, b does the system $DX = \begin{bmatrix} 5 \\ -5 \\ 7 \\ 0 \end{bmatrix}$ have a unique solution?

-4

$$\left[\begin{array}{cccc|c} 4 & 2 & b & a & 5 \\ -4 & -2 & -2 & 2 & -5 \\ 2 & 0.5b & 0 & 1 & 7 \\ 0 & 0 & 0 & 10 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cccc|c} 1 & 0.5 & b/4 & a/4 & 5/4 \\ -4 & -2 & -2 & 2 & -5 \\ 2 & 0.5b & 0 & 1 & 7 \\ 0 & 0 & 0 & 10 & 0 \end{array}\right]$$

$4R_1 + R_2$

$-2R_1 + R_3 \to$

$$\left[\begin{array}{cccc|c} 1 & 0.5 & b/4 & a/4 & 5/4 \\ 0 & 0 & b-2 & a+2 & 0 \\ 0 & 0.5b-1 & -b/2 & -a/2+1 & 9/2 \\ 0 & 0 & 0 & 10 & 0 \end{array}\right] \xrightarrow{\frac{1}{10}R_4} \left[\begin{array}{cccc|c} 1 & 0.5 & b/4 & a/4 & 5/4 \\ 0 & 0 & b-2 & a+2 & 0 \\ 0 & 0.5b-1 & -b/2 & -a/2+1 & 9/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(ii) For what values of a, b will D be singular(non-invertible)?

$$\left[\begin{array}{cccc|c} 1 & 0.5 & b/4 & a/4 & 5/4 \\ 0 & 0 & b-2 & a+2 & 0 \\ 0 & 0.5b-1 & -b/2 & -a/2+1 & 9/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0.5 & b/4 & a/4 & 5/4 \\ 0 & 0.5b-1 & -b/2 & -a/2+1 & 9/2 \\ 0 & 0 & b-2 & a+2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$-\det$

behind page

$1 \times (0.5b-1) \times (b-2) \times 1 = 0$

For $\boxed{b=2}$

a it doesn't effect

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & C \end{array} \\
\left[\begin{array}{cccc|c}
1 & 0.5 & b/4 & a/4 & 5/4 \\
0 & 0 & b-2 & a+2 & 0 \\
0 & 0.5b-1 & -b/2 & -a/2+1 & 9/2 \\
0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$x_1 + 0.5x_2 + \frac{b}{4}x_3 + \frac{a}{4}x_4 = 5/4$$

$$(b-2)x_3 + (a+2)x_4 = 0$$

$$(0.5b-1)x_2 - \frac{b}{2}x_3 + \left(-\frac{a}{2}+1\right)x_4 = 9/2$$

$$x_4 = 0$$

it cant be because the system is inconsistant

(V) find the matrix

$$
\left[\begin{array}{ccc|ccc}
1.5 & 1 & 0 & 1 & 0 & -3 \\
1 & 1 & 0 & 0 & 1 & -2 \\
1 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
\begin{array}{l} 2R_3 + R_2 \to R_2 \\ \hline 3R_3 + R_1 \to R_1 \end{array}
$$

$$
\left[\begin{array}{ccc|ccc}
4.5 & 1 & 3 & 1 & 0 & 0 \\
3 & 1 & 2 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

A $\qquad$ I_n

so A is =
$$
\begin{bmatrix}
4.5 & 1 & 3 \\
3 & 1 & 2 \\
1 & 0 & 1
\end{bmatrix}
$$

QUESTION 3. (30 points) Let A be a 4×4 matrix. Given

$$A \underbrace{3R_1} A_1 \underbrace{R_3 \leftrightarrow R_2} A_2 \underbrace{-15R_2} A_3 \underbrace{-4R_1 + R_2 \to R_2} A_4 = \begin{bmatrix} 3 & 2 & 2 & -4 \\ 0 & 5 & 3 & -2 \\ 0 & 0 & 9 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

(i) Find $det(A)$.

$det(A_4) = det(A_3)$, $det(A_3) = -15\,det(A_2)$, $det(A_2) = -det(A_1)$

$det(A_1) = 3\,det(A)$

$det(A_4) = 3 \times 5 \times 9 \times 8 = 1080$

$det(A_4) = -15(3(-det(A)))$

$det(A_4) = 45\,det(A) \implies det(A) = \frac{det(A_4)}{45} = \frac{1080}{45} = \boxed{2}$

(ii) Find $det(-2A_2)$

$det(A_4) = -15\,det(A_2)$

$det(A_2) = \frac{det(A_4)}{-15} = \frac{1080}{-15} = -72$

$-2^n\,det(A_2)$

$-2^2 \cdot -72$

$4 \cdot -72 \quad \boxed{= -288}$

(iii) Find a matrix B such that $BA = A_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \underbrace{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{B}$$

$B \cdot A = A_2$

(iv) Find $det(A^2A_2)$

$= det(A)^2 \cdot det(A_2)$

$= (24)^2 \cdot -72 = \boxed{-41472}$

(v) Find $det((A^{-1}A_4)^T)$

transpose doesnt effect the det so $det(A^{-1}) \cdot det(A_4)$

$= \frac{1}{24} \cdot 1080 = \boxed{45}$

(vi) Find the $(3,4)$-entry of A_4^{-1}

$(3,4)$-entry of $A_4^{-1} = \frac{C_{43}}{det(A_4)} = \frac{(-1)^7 \begin{bmatrix} 3 & 2 & -4 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix}}{det(A_4)}$

$= \frac{-1(3 \times 5 \times 2)}{1080} = \frac{-30}{1080} \boxed{= \frac{-1}{36}}$

$2\begin{bmatrix}4\\1\end{bmatrix} + -1\begin{bmatrix}7\\ \end{bmatrix}$

QUESTION 4. (12 points) (b) Find a 2×2 matrix A such that $A\begin{bmatrix}6&7\\1&4\end{bmatrix}+3I_2 = 2A+\begin{bmatrix}4&2\\0&2\end{bmatrix}$

$$A\begin{bmatrix}6&7\\1&4\end{bmatrix} - 2A = \begin{bmatrix}4&2\\0&2\end{bmatrix} - \begin{bmatrix}3&0\\0&3\end{bmatrix}$$

$$A\left(\begin{bmatrix}6&7\\1&4\end{bmatrix} - \begin{bmatrix}2&0\\0&2\end{bmatrix}\right) = \begin{bmatrix}1&2\\0&-1\end{bmatrix}$$

$$A\begin{bmatrix}4&7\\1&2\end{bmatrix} = \begin{bmatrix}1&2\\0&-1\end{bmatrix} \Longrightarrow A\begin{bmatrix}4&7\\1&2\end{bmatrix}\begin{bmatrix}2&-7\\-1&4\end{bmatrix} = \begin{bmatrix}1&2\\0&-1\end{bmatrix}\begin{bmatrix}2&\\-1&\end{bmatrix}$$

behind

QUESTION 5. (16 points) Solve the following system:

$x_2 + x_3 - x_4 = 2$
$x_1 - x_2 - x_5 = 4$
$2x_1 + x_3 - 2x_5 = 2$

Then give me one numerical solution to the system.

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & c \end{array}$$
$$\left[\begin{array}{ccccc|c} 0&1&1&-1&0&2\\ 1&-1&0&0&-1&4\\ 2&0&1&0&-2&2 \end{array}\right] \xrightarrow{R_1+R_2\to R_2} \left[\begin{array}{ccccc|c} 0&1&1&-1&0&2\\ 1&0&1&-1&-1&6\\ 2&0&1&0&-2&2 \end{array}\right]$$

$-2R_2+R_3$

$$\left[\begin{array}{ccccc|c} 0&1&1&-1&0&2\\ 1&0&1&-1&-1&6\\ 0&0&-1&2&0&-10 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccccc|c} 0&1&1&-1&0&2\\ 1&0&1&-1&-1&6\\ 0&0&1&-2&0&10 \end{array}\right]$$

$-R_3+R_1\to R_1$
$-R_3+R_2\to R_2$

$$\begin{array}{cccccc} x & x_2 & x_3 & x_4 & x_5 & c \end{array}$$
$$\left[\begin{array}{ccccc|c} 0&1&0&1&0&-8\\ 1&0&0&1&-1&-4\\ 0&0&1&-2&0&10 \end{array}\right]$$

completely reduced

read

$x_2 + x_4 = -8$
$x_1 + x_4 - x_5 = -4$
$x_3 - 2x_4 = 10$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

Free variables x_4 and x_5

$x_2 = -8 - x_4$
$x_1 = -4 \cancel{-x_4} + x_5$
$x_3 = 10 \cancel{+2x_4}$
$x_4, x_5 \in \mathbb{R}$

by adding 3 equations

$x_2 = -8$
$x_1 = -4 + x_5$
$x_3 = 10$

$$AI_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + -1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$-7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}$$